Komp. Fonk. Teo. Giriş YAZ DÖNEMİ Arasınav Soruları ve Çözümleri

1) $|z|=1$ ise $\left|\dfrac{a.z+b}{\bar{b}.z+\bar{a}}\right|=1$ $(a,b\in\mathbb{C})$ olduğunu gösteriniz.

Çözüm: $|z|^2=z.\bar{z}$ olduğundan $|z|=1 \Rightarrow \bar{z}=\dfrac{1}{z}$ olur.

$$\left|\frac{a.z+b}{\bar{b}.z+\bar{a}}\right| = \frac{a.z+b}{\bar{b}+\frac{\bar{a}}{z}}\cdot\frac{1}{z} = \frac{a.z+b}{\bar{b}+\bar{a}.\bar{z}}\cdot\frac{1}{z} = \frac{a.z+b}{\bar{b}.z+\bar{a}.\bar{z}.z} = \frac{a.z+b}{\bar{b}z+\bar{a}|z|^2} =$$

$$= \frac{a.z+b}{\bar{b}.z+\bar{a}} \Rightarrow$$

$$\left|\frac{a.z+b}{\bar{b}z+\bar{a}}\right| = \left|\frac{a.z+b}{\bar{b}.z+\bar{a}}\right| = \left|\frac{az+b}{\overline{az+b}}\cdot\frac{1}{z}\right| = \frac{|az+b|}{|\overline{az+b}|}\cdot\frac{1}{|z|} = 1 \quad (|z|=|\bar{z}|)$$

bulunur.

2) $z\in\mathbb{C}$, $|z|=r>0$ olan her $z\in\mathbb{C}$ için $Re\,z=\frac{1}{2}\left(z+\frac{r^2}{z}\right)$ ve $Im\,z=\frac{1}{2i}\left(z-\frac{r^2}{z}\right)$ olduğunu göst.

Çözüm: $z=x+iy \Rightarrow Re\,z=x,\ Im\,z=y \Rightarrow |z|^2=r^2=x^2+y^2$

$$\frac{1}{2}(x+iy)+\frac{1}{2}\left(\frac{x^2+y^2}{x+iy}\right) = \frac{1}{2}(x+iy)+\frac{1}{2}\frac{(x^2+y^2)(x-iy)}{x^2+y^2}$$

$$=\frac{1}{2}(x+iy)+\frac{1}{2}(x-iy) = x = Re\,z \Rightarrow$$

$Re\,z=\frac{1}{2}\left(z+\frac{r^2}{z}\right)$ bulunur.

$$\frac{1}{2i}\left(z-\frac{r^2}{z}\right) = \frac{1}{2i}\left(\frac{z^2-r^2}{z}\right) = \frac{1}{2i}\left(\frac{x^2-y^2+2ixy-x^2-y^2}{x+iy}\right) =$$

$$=\frac{1}{2i}\left(\frac{-2y^2+2ixy}{x+iy}\right) = \frac{1}{i}y\left(\frac{-y+ix}{x+iy}\right) = y.\left(\frac{-y+ix}{-y+ix}\right) = y = Im\,z$$

$\Rightarrow Im\,z=\frac{1}{2i}\left(z-\frac{r^2}{z}\right)$ bulunur.

3) $\cos z$ nin reel ve sanal kısımlarını bulunuz.
$|\cos z|^2 = \cos^2 x + \sinh^2 y$ olduğunu gösteriniz.

<u>Çözüm:</u> $z = x + iy$, $x, y \in \mathbb{R}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right)$$

$$= \frac{1}{2}\left(e^{ix-y} + e^{-ix+y}\right)$$

$$= \frac{1}{2}\left[e^{-y}.(\cos x + i\sin x) + e^{y}.(\cos x - i\sin x)\right]$$

$$= \cos x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\sin x\left(\frac{e^{-y} - e^{y}}{2}\right)$$

$$= \cos x.\cosh y - i\sin x.\sinh y$$

$\operatorname{Re}(\cos z) = \cos x.\cosh y$, $\operatorname{Im}(\cos z) = -\sin x.\sinh y$.

$$|\cos z|^2 = \cos^2 x.\cosh^2 y + \sin^2 x.\sinh^2 y$$

$$= \cos^2 x.\cosh^2 y + (1 - \cos^2 x).\sinh^2 y$$

$$= \cos^2 x.\cosh^2 y + \sinh^2 y - \cos^2 x.\sinh^2 y$$

$$= \cos^2 x\,(\underbrace{\cosh^2 y - \sinh^2 y}_{1}) + \sinh^2 y$$

$$= \cos^2 x + \sinh^2 y \quad \text{bulunur.}$$

4) $(-2\sqrt{3} - 2i)^{1/4}$, $(1+i)^{1+i}$ sayılarının değerlerini $a+ib$ formunda yazınız.

<u>Çözüm:</u>

$$(1+i)^{1+i} = e^{(1+i)\log(1+i)} = e^{(1+i)\left[\ln|1+i| + i\arg(1+i)\right]}$$

$$= e^{(1+i).\left[\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)\right]} = e^{\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right) + i\ln\sqrt{2} - \left(\frac{\pi}{4} + 2k\pi\right)}$$

$$= e^{\overbrace{\ln\sqrt{2} - \left(\frac{\pi}{4} + 2k\pi\right)}^{x} + i\overbrace{\left[\left(\frac{\pi}{4} + 2k\pi\right) + \ln\sqrt{2}\right]}^{y}} = e^{x+iy} = e^{x}.(\cos y + i\sin y)$$

$$(1+i)^{1+i} = e^{x}.\cos\left(\frac{\pi}{4} + 2k\pi + \ln\sqrt{2}\right) + i\,e^{x}.\sin\left(\frac{\pi}{4} + 2k\pi + \ln\sqrt{2}\right)$$

$$a = e^{\ln\sqrt{2} - \left(\frac{\pi}{4} + 2k\pi\right)}.\cos\left(\frac{\pi}{4} + 2k\pi + \ln\sqrt{2}\right), \quad b = e^{\ln\sqrt{2} - \left(\frac{\pi}{4} + 2k\pi\right)}.\sin\left(\frac{\pi}{4} + 2k\pi + \ln\sqrt{2}\right)$$

$(-2\sqrt{3}-2i)^{1/4} = a+ib$ o.ş. $a,b\in\mathbb{R}$ bulmalıyız.

$w = -2\sqrt{3}-2i$, $|w| = (-2\sqrt{3})^2+(-2)^2 = 4\cdot 3+4 = 16$, $|w| = 16$

$\cos\theta = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$, $\sin\theta = \frac{-2}{4} = -\frac{1}{2}$, $\theta = \arg w = \pi+\frac{\pi}{6} = \frac{7\pi}{6}$

$$z_k = w^{1/4} = 16^{1/4}\cdot\left[\cos\frac{\frac{7\pi}{6}+2k\pi}{4} + i\sin\frac{\frac{7\pi}{6}+2k\pi}{4}\right], \quad k=0,1,2,3,4$$

$$z_0 = 2\cdot\left[\cos\frac{7\pi}{24} + i\sin\frac{7\pi}{24}\right], \quad z_2 = 2\left[\cos\frac{31\pi}{24} + i\sin\frac{31\pi}{24}\right]$$

$$z_1 = 2\cdot\left[\cos\frac{19\pi}{24} + i\sin\frac{19\pi}{24}\right], \quad z_3 = 2\left[\cos\frac{43\pi}{24} + i\sin\frac{43\pi}{24}\right]$$

- . . .

(2) $2z^2-(2+5i)z-2+i=0$ denklemini çözünüz.

$$\Delta = b^2-4ac = (2+5i)^2-4\cdot 2\cdot(-2+i) = 4+20i-25+16-8i$$
$$= -5+12i$$

$$z_{1,2} = \frac{2+5i \mp \sqrt{-5+12i}}{4}$$

$\sqrt{-5+12i}$ yi bulalım.

$$\sqrt{-5+12i} = \mp\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} + \mu\cdot i\cdot\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}} =$$

$$= \mp\sqrt{\frac{-5+\sqrt{25+144}}{2}} + i\sqrt{\frac{5+\sqrt{25+144}}{2}}$$

$$= \mp\sqrt{\frac{-5+13}{2}} + i\sqrt{\frac{+5+13}{2}} = \mp(2+i3)$$

$$z_1 = \frac{2+5i+2+3i}{4} = \frac{4+8i}{4} = 1+2i \Rightarrow \boxed{z_1 = 1+2i}$$

$$z_2 = \frac{2+5i-2-3i}{4} = \frac{0+2i}{4} \Rightarrow z_2 = \frac{i}{2} \qquad \zeta = \left\{1+2i, \frac{i}{2}\right\}$$

⑥ $f(z)=\dfrac{\bar{z}-1}{\operatorname{Arg} z}+\dfrac{3}{\cos z-\frac{1}{2}}$ fonk. tanım kümesini bulunuz.

Çözüm: $f_1(z)=\dfrac{\bar{z}-1}{\operatorname{Arg} z}$, $f_2(z)=\dfrac{3}{\cos z-\frac{1}{2}}$

$D_f=Df_1\cap Df_2$

$D(f_1)=\mathbb{C}-\{z\in\mathbb{C} : \operatorname{Arg} z=0,\ z\neq 0\}$. $\operatorname{Arg} z=0$ olan $z=2k\pi$ dir.
$z=2k\pi,\ k\neq 0,\ k\in\mathbb{Z}$.

$z=|z|.(\cos(\operatorname{Arg} z)+i\sin(\operatorname{Arg} z))$
$=|z|.(\cos 0+i\sin 0)$

$z=|z| \iff z\in\mathbb{R}^+$ olur. pozitif reel sayılar.

$D(f_1)=\mathbb{C}-\{z=x+iy : x\geq 0,\ y=0\}$

$D(f_2)=\{z\in\mathbb{C} : \cos z-\frac{1}{2}\neq 0\}=\mathbb{C}-\{z\in\mathbb{C} : \cos z=\frac{1}{2}\}$.

$\cos z=\frac{1}{2} \iff \dfrac{e^{iz}+e^{-iz}}{2}=\dfrac{1}{2} \iff e^{iz}+e^{-iz}=1$ $t=e^{iz}$,

$t+\frac{1}{t}=1 \iff t^2-t+1=0$ $\Delta=(-1)^2-4.1.1=-3=3i^2$

$t_{1,2}=\dfrac{1\mp i\sqrt{3}}{2} \Rightarrow e^{iz}=\dfrac{1\mp i\sqrt{3}}{2} \Rightarrow iz=\log\left(\dfrac{1\mp i\sqrt{3}}{2}\right)$

$iz=\log\left(\frac{1}{2}\mp i\frac{\sqrt{3}}{2}\right)=\ln\left|\frac{1}{2}\mp i\sqrt{3}\right|+i\arg\left(\frac{1}{2}\mp i\frac{\sqrt{3}}{2}\right)$

$iz=\ln\left|\frac{1}{2}+i\frac{\sqrt{3}}{2}\right|+i\arg\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$, $iz=\ln\left|\frac{1}{2}-i\frac{\sqrt{3}}{2}\right|+i\arg\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$

$\left|\frac{1}{2}\mp i\frac{\sqrt{3}}{2}\right|=1$, $\arg\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}+2k\pi$, $\arg\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}+2k\pi$

$z=\frac{1}{i}\left(\ln 1+i\left(\frac{\pi}{3}+2k\pi\right)\right)=\frac{1}{i}.i\left(\frac{\pi}{3}+2k\pi\right)=\frac{\pi}{3}+2k\pi$, $k\in\mathbb{Z}$

$z=\frac{1}{i}\left[\ln 1+i\left(-\frac{\pi}{3}+2k\pi\right)\right]=\frac{1}{i}.i\left(-\frac{\pi}{3}+2k\pi\right)=-\frac{\pi}{3}+2k\pi$, $k\in\mathbb{Z}$

$D(f_2)=\mathbb{C}-\{\frac{\pi}{3}+2k\pi,\ -\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}\}$.

7) $f(z)=\frac{1}{z}$ fonksiyonu verildiğinde hangi eğriler bu dönüşümle w_düzleminde $u=$sabit, $v=$sabit doğrulara dönüşür.

Çözüm: $w=f(z)=\frac{1}{z}=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}=u+iv$

$u=\frac{x}{x^2+y^2}$, $v=\frac{-y}{x^2+y^2}$ $\Leftrightarrow x^2+y^2=\frac{x}{u} \Rightarrow x^2+y^2-\frac{x}{u}=0$

$\Rightarrow \left(x-\frac{1}{2u}\right)^2+y^2=\frac{1}{4u^2}$ ve

$x^2+y^2=\frac{-y}{v} \Rightarrow x^2+y^2+\frac{y}{v}=0 \Rightarrow x^2+\left(y+\frac{1}{2v}\right)^2=\frac{1}{4v^2}$ bulunur.

$u=$sabit$=k$ olsun. $\left(x-\frac{1}{2k}\right)^2+y^2=\frac{1}{(2k)^2} \Leftrightarrow M\left(+\frac{1}{2k},0\right), r=\left|\frac{1}{2k}\right|$

olan çember belirtir. Yani $\left|z-\frac{1}{2k}\right|=\left|\frac{1}{2k}\right|$ olur.

$v=$sabit$=k$ olsun. $x^2+\left(y+\frac{1}{2v}\right)^2=\left(\frac{1}{2v}\right)^2$, $M\left(0,-\frac{1}{2v}\right), r=\left|\frac{1}{2v}\right|$

olan çember belirtir. Yani $\left|z+\frac{1}{2k}\right|=\left|\frac{1}{2k}\right|$ ol. O halde

$\left|z-\frac{1}{2k}\right|=\left|\frac{1}{2k}\right|$, $\left|z+\frac{1}{2k}\right|=\left|\frac{1}{2k}\right|$ çemberleri $u=$sabit$=v$

doğrularına dönüşür.